STA238 Tutorial 6

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2023-08-01

1 Announcements

- You can upload your work on Crowdmark from the end of the tutorial session to 5pm Friday of that week.
- All questions must be solved using RStudio.

2 Recall: Last tutorial

Last tutorial: based on a random sample, and two measurements for a subject (or two paired populations) we conducted a **paired t-test**, which was a one-tailed test. In addition, we looked at the assumptions behind a paired t-test

Main takeaways:

1. A paired t-test is used for subjects who have a measurement before and after a certain event, or comparing between two paired populations.
2. The rejection region for the paired t-test comes from a t-value. However, if the **sample size is large enough, then one can use the z-values by the CLT.**
3. The paired t-test relies on two assumptions:

- The sample was randomly collected.
- The **differences of the measurements/populations follow a normal distribution (can check this by using a QQ-plot)**

3 Tutorial activity

We want to:

1. State and understand the method of moments.
2. Compute the moments for a Gamma distribution.
3. Use the method of moments to obtain estimators for the parameters of a Gamma distribution

3.1 Method of Moments

Generally, the intuition behind the method of moments is that it will be a method used to propose estimators for parameters of a probability distribution (i.e. how can we estimate μ and σ^2 for a normal distribution?)

Given a probability distribution $f(x;\theta)$ of a random variable X with parameters θ_1 and θ_2 which characterize it, if we can write first two moments of the distribution as a function of θ_i:

$$\mu_1 = E(X) = g_1(\theta_1, \theta_2)$$
$$\mu_2 = E(X^2) = g_2(\theta_1, \theta_2)$$

If we let $\hat{\mu}_1$ and $\hat{\mu}_2$ be the sample first and second moment, i.e:

$$\hat{\mu}_j = \frac{1}{n}\sum_{i=1}^{n} x_i^j$$

Then, the method of moments estimator $\hat{\theta}_1$ and $\hat{\theta}_2$ will be the solution to the following system of equations:

$$\hat{\mu}_1 = g_1(\hat{\theta}_1, \hat{\theta}_2)$$
$$\hat{\mu}_2 = g_2(\hat{\theta}_1, \hat{\theta}_2)$$

These estimators are usually consistent, but can be biased.

3.2 Application: Estimating the parameters from the Gamma distribution

The Gamma distribution has the following pdf, if $Y \sim \text{Gamma}(\alpha, \beta)$:

$$f(y) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} & 0 < y < \infty \\ 0 \quad \text{otherwise} \end{cases}$$

If we have n observations Y_i from a Gamma distribution with parameters α, β, we can compute:

$$E(Y) = \int_0^\infty y \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} dy = \int_0^\infty \frac{\beta}{\Gamma(\alpha)\beta^{\alpha+1}} y^{\alpha} e^{-y/\beta} dy$$

Recall that from the definition of the Gamma function (using the change of variables $y/\beta = z$):

$$\Gamma(\alpha) = \int_0^\infty \frac{y^{\alpha-1}}{\beta^\alpha} e^{-y/\beta} dy = \int_0^\infty z^{\alpha-1} e^{-z} dz$$

3.2.1 Computing moments of the Gamma distribution

Hence:

$$E(Y) = \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \frac{1}{\beta^{\alpha+1}} y^\alpha e^{-y/\beta} dy = \frac{\beta\Gamma(\alpha+1)}{\Gamma(\alpha)} = \beta\alpha$$

Remark: We have used the property $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, which can be verified by using integration by parts.

Similarly, one finds:

$$E(Y^2) = \int_0^\infty y^2 \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} dy = \int_0^\infty \frac{\beta^2}{\Gamma(\alpha)\beta^{\alpha+2}} y^{\alpha+1} e^{-y/\beta} dy$$

Then, one obtains:

$$E(Y^2) = \frac{\beta^2}{\Gamma(\alpha)}\Gamma(\alpha+2) = \beta^2(\alpha+1)\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \beta^2(\alpha+1)\alpha$$

3.2.2 Applying this to the Method of Moments estimation

One will have:

$$\hat{\mu}_1 = \overline{Y} = \hat{\beta}\hat{\alpha}$$
$$\hat{\mu}_2 = \overline{Y^2} = \hat{\beta}^2(\hat{\alpha}+1)\hat{\alpha}$$

Then, our **goal is to solve the simultaneous equations for $\hat{\alpha}$ and $\hat{\beta}$.**

We have:

$$\overline{Y^2} = \hat{\beta}^2(\hat{\alpha}+1)\hat{\alpha} = (\overline{Y})^2 + \hat{\beta}\overline{Y}$$
$$\Rightarrow \hat{\beta} = \frac{\overline{Y^2} - (\overline{Y})^2}{\overline{Y}}$$

Rearranging the above:

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}{n\overline{Y}}$$

Now, one can solve for $\hat{\alpha}$:

$$\overline{Y} = \hat{\alpha}\hat{\beta} = \hat{\alpha}\frac{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}{n\overline{Y}}$$

We get:

$$\hat{\alpha} = \frac{n(\overline{Y})^2}{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}$$

Both $\hat{\alpha}$ and $\hat{\beta}$ will be the method of moment estimators for the parameters of the Gamma distribution.